# A COMPARATIVE STUDY OF LQR AND SLIDING CONTROL FOR BALL AND BEAM SYSTEM

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**Abstract:** In this paper, we will survey two controllers, a linear control (LQR) and sliding mode control (SMC) on a central axis ball and beam (B&B) – a single input-multiple output (SIMO) system through simulation and experiment. In experiment, we present a hardware platform using STM32F407 for this model. Beside simulation results, the results in experiment prove the effectiveness of both methods in different cases. Thence, through this research, the advantages and disadvantages of the two controllers are demonstrated for designers to select in suitable cases.

**Keywords:** LQR, sliding mode control, ball and beam, SIMO system.

### 1. Introduction

In automation control, various control algorithms are extensively studied to aid designers in selecting suitable controllers for systems and tuning them appropriately for each specific application. The linear Proportional-Integral-Derivative (PID) algorithm [1] is known for its simple structure, but parameter tuning relies solely on trial and error, posing challenges to system stability and lacking mathematical guarantees. The Linear Quadratic Regulator (LQR) algorithm [2] addresses this limitation by solving the Riccati equation, ensuring mathematical stability. However, its stability is confined to a limited "neighborhood" around the operating point, risking instability if this neighborhood is exceeded. Therefore, achieving local stability using these algorithms alone remains incomplete.

In the case of trajectory tracking control, both linear PID and LQR algorithms are unsuitable due to their inherent simplicity, primarily suited for local stability. A feasible solution lies in non-linear algorithms, which can ensure mathematical stability through Lyapunov stability standards [3] and guarantee a working space across the entire system. Among non-linear algorithms, SMC [4][5] is widely employed. In this project, our approach is to apply both LQR and SMC algorithms to a typical SIMO system - B&B [6] to compare the pros and cons as well as the stability quality of the linear LQR and non-linear SMC controllers. Searching for parameters for the SMC map is based on a genetic algorithm (GA) [7]

### 2. Mathematical Model

B&B mathematical model is shown in Fig. 1 below

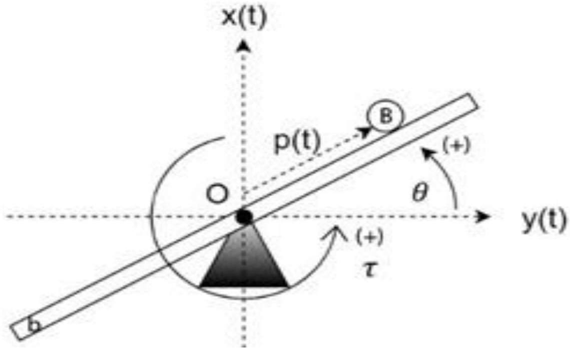


**Fig. 1.** Mathematical model of B&B [6]

The system consists of a horizontal bar (beam), a ball, a DC motor, the beam is wrapped with electric wire and supplied with a voltage to read the ball's position using ADC and determine the tilt angle. of the bar using Encoder. We control the position of the ball on the bar by changing the angle of inclination of the beam compared to the horizontal through a DC motor. This is a highly nonlinear system. Under the influence of gravity, with a small tilt angle of the bar, the ball will roll very quickly and be difficult to maintain in a balanced position. The stability of the system depends not only on the structure or system parameters but also on the input signal - the voltage supplied to the motor. The dynamic equation of B&B can be written as follows:

$$\ddot{p} = \frac{m_B p\dot{\theta}^2 - m_B g \sin\theta}{m_B + \dfrac{J_B}{R^2}} \tag{1}$$

$$\ddot{\theta} = \frac{K_t \dfrac{e - K_b\dot{\theta}}{R_m} - 2m_B p\dot{p}\dot{\theta} - m_B g p \cos\theta}{m_B p^2 + J_b} \tag{2}$$

where $J_B = \frac{2}{5} m_B R^2$ ; $J_b = \frac{1}{12} m_B L_b{}^2$; $p(t)$ is position of ball relative to center of beam (m), $m_B$ is mass of ball (kg), $m_b$ is mass of beam (kg), $L_b$ is length of beam (m), $R$ is diameter of ball (m), theta is rotation angle of beam relative to horizontal (rad), $K_t$ is torque constant (Nm/A), $K_b$ is reaction constant (V/(rad/s)), $R_m$ is motor resistance ($\Omega$), $J_B$ is moment of inertia of ball ($kgm^2$), $J_b$ is moment of inertia of beam ($kgm^2$), $e$ is voltage supplied to motor (V).

## 3. Controller Designing

### 3.1. LQR Controller

We set:

$$x = \begin{bmatrix} p & \dot{p} & \theta & \dot{\theta} \end{bmatrix}^T = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \tag{3}$$

The system mathematical equations can be written as:

$$\dot{x}_1 = x_2 = f_1(x,e) \tag{4}$$

$$\dot{x}_2 = \frac{m_B x_1 x_4^2 - m_B g \sin x_3}{m_B + \frac{J_B}{R^2}} = f_2(x,e) \tag{5}$$

$$\dot{x}_3 = x_4 = f_3(x,e) \tag{6}$$

$$\dot{x}_4 = \frac{\begin{pmatrix} K_t e - K_t K_b x_4 - 2m_B R_m x_1 x_2 x_4 \\ -R_m m_B g x_1 \cos x_3 \end{pmatrix}}{\left(m_B x_1^2 + J_b\right) R_m} = f_4(x,e) \tag{7}$$

From (4), (5), (6), (7), we see that equations system has the form

$$\dot{x} = f(x,e) \tag{8}$$

where: $f = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}^T$

If we assume that system just operates around working point:

$$x_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, e = 0 \tag{9}$$

, nonlinear B&B can be accepted to be equivalent to a linear form:

$$\dot{x} = Ax + Be \tag{10}$$

**Tab. 1.**System parameters

| Parameters | Descriptions | Value | Units |
|---|---|---|---|
| $K_t$ | Coefficient of motor | 0.06494 | N |
| $K_b$ | Coefficient of motor | 0.06494 | V/rad/s |
| $R_m$ | Coefficient of motor | 6.83572 | $\Omega$ |
| $m_B$ | Mass of ball | 0.06 | Kg |
| $m_b$ | Mass of beam | 0.38 | Kg |
| $R$ | Radius of ball | 0.01 | m |
| $L_b$ | Length of beam | 0.35 | m |

With the actual values of the model, at the working point, the results of matrices A, B can be calculated as:

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}_{x=0,e=0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -7.01 & 0 \\ 0 & 0 & 0 & 1 \\ -130.93 & 0 & 0 & -0.18 \end{bmatrix} \tag{11}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial e} \\ \frac{\partial f_2}{\partial e} \\ \frac{\partial f_3}{\partial e} \\ \frac{\partial f_4}{\partial e} \end{bmatrix}_{x=0,e=0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2.82 \end{bmatrix} \tag{12}$$

LQR control signal is

$$u = -Kx \tag{13}$$

where $K = \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \end{bmatrix}$ is control matrix, K is calculated from solving Ricatti equation. This work is complicated. Thence, Matlab software provides tools to do this

$$K = dlqr\left(A_d, B_d, Q, R, T\right) \tag{14}$$

In Matlab, T is sample-time, Ad and Bd are calculated by using command:

$$\begin{bmatrix} A_d & B_d \end{bmatrix} = c2d\left(A, B, T\right) \tag{15}$$

We choose through GA:

$$Q = \begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 280 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}, R = 0.01 \tag{16}$$

$$\rightarrow K = \begin{bmatrix} -110.45 & -100.85 & 190.6 & 12.83 \end{bmatrix} \tag{17}$$

**Tab. 2.** Blocks in the LQR simulation program in Fig. 2

| | |
|---|---|
| 1 | \|Set points |
| 2 | Data storage blocks |
| 3 | Data display block |
| 4 | Blocks help stop the simulation |
| 5 | B&B description block |
| 6 | Sum block |
| 7 | LQR controller |

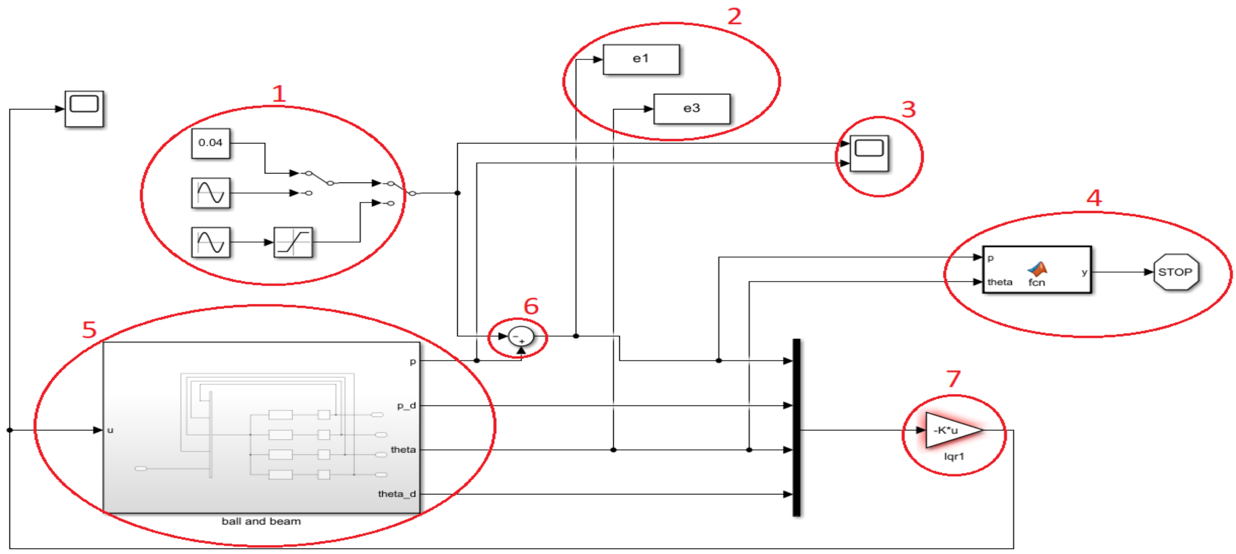


**Fig. 2.** LQR controller

### 3.2. Sliding Controller

From equations (4), (5), (6) and (7), System of state equations is derived as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(x) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2(x) + g_2(x)u \end{cases} \tag{18}$$

Set the sample state for the system:

$$x_d = \begin{bmatrix} x_{1d} & x_{2d} & x_{3d} & x_{4d} \end{bmatrix} \tag{19}$$

In there, $x_{1d}, x_{2d}, x_{3d}, x_{4d}$ are the placed orbits of state variables $x_1, x_2, x_3, x_4$ and $x_{3d} = x_{4d} = 0$, $x_{1d}$ is a function of time depends on the trajectory we want the ball to have follow

Set:

$$\begin{cases} e_p = x_1 - x_{1d} \\ e_\theta = x_3 \end{cases} \tag{20}$$

**Select sliding surface:**

$$S_1 = \dot{e}_\theta + c_1 e_\theta + c_2 e_p + c_3 \dot{e}_p \tag{21}$$

$$= x_4 + c_1 x_3 + c_2 (x_1 - x_{1d}) + c_3 (x_2 - x_{2d}) \tag{22}$$

$$\rightarrow \dot{S}_1 = \dot{x}_4 + c_1 \dot{x}_3 + c_2 (\dot{x}_1 - \dot{x}_{1d}) + c_3 (\dot{x}_2 - \dot{x}_{2d})$$

$$= \begin{pmatrix} g_2(x)u + f_2(x) + c_1 x_4 \\ +c_2 (x_2 - x_{2d}) + c_3 (f_1(x) - \dot{x}_{2d}) \end{pmatrix} \tag{23}$$

Let:

$$h(x) = \begin{pmatrix} f_2(x) + c_1 x_4 + c_2 (x_2 - x_{2d}) \\ +c_3 (f_1(x) - \dot{x}_{2d}) \end{pmatrix} \tag{24}$$

Substituting (24) into (23), we obtain:

$$\dot{S}_1 = g_2(x)u + h(x) \tag{25}$$

**Choose Lyapunov function:**

$$V = \frac{1}{2} S_1^2 \tag{26}$$

$$\dot{V} = S_1 \dot{S}_1 \text{ (need to be selected <0)} \tag{27}$$

Select:

$$\dot{S}_1 = -\eta sign(S_1) \text{ with } \eta > 0 \tag{28}$$

$$\rightarrow \dot{V} = -\eta S_1 sign(S_1) < 0 \tag{29}$$

Substituting (28) into (25), we obtain:

$$g_2(x)u + h(x) = -\eta sign(S_1) \tag{30}$$

$$\rightarrow u = \frac{-\eta sign(S_1) - h(x)}{g_2(x)} \tag{31}$$

The parameters of SMC found from GA are:

$$c_1 = 25.4, c_2 = -42.61, c_3 = -47.61, nuy = 12.78. \tag{32}$$

**Tab. 3**. SMC control explanation for Fig. 3

| | |
|---|---|
| 1 | Set points |
| 2 | Sum block |
| 3 | Derivative block |
| 4 | The parameters of SMC |
| 5 | SMC controller |
| 6 | Data display block |
| 7 | B&B description block |
| 8 | Blocks help stop the simulation |
| 9 | Data storage block |

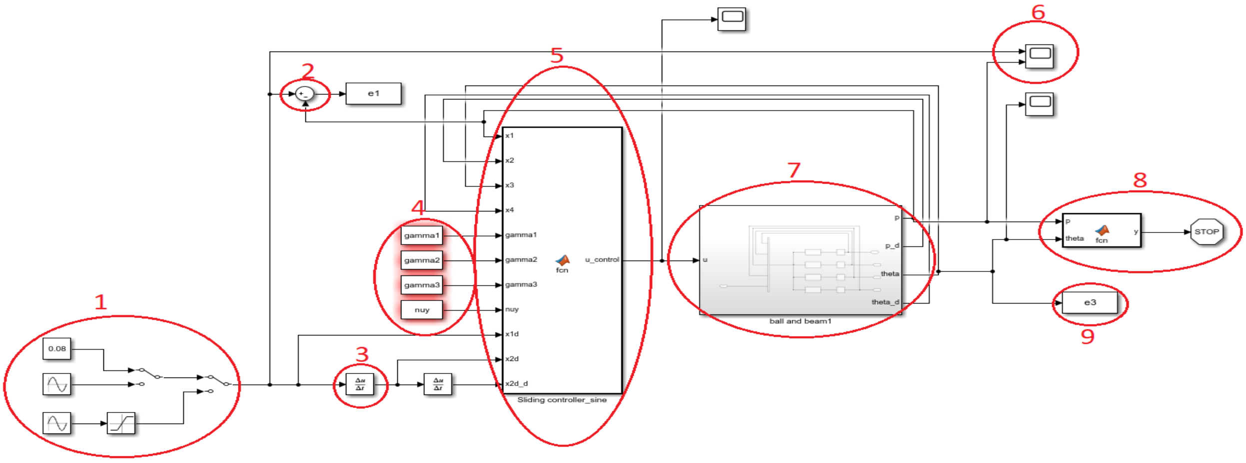

**Fig. 3.** SMC control simulation

## 4. Simulation

**Case 1:** Stable around the equilibrium position, $x_{1d} = 0$

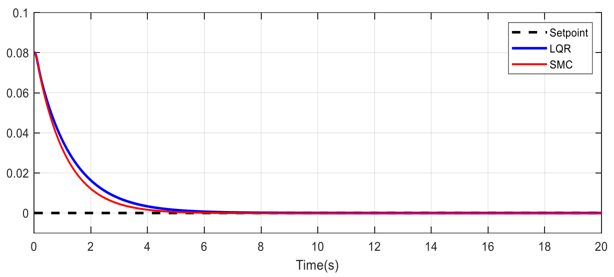

**Fig. 4**. Position of ball (m)

In Fig. 4, although both methods have a setup time of 6 seconds, the smaller steady-state error of LQR compared to SMC indicates that LQR is more stable around the equilibrium position than SMC.

**Case 2:** Worksite changes corresponding to the change in set value, $x_{1d} = 0.08(m)$

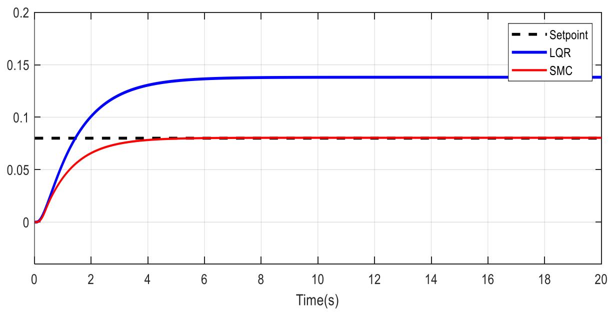

**Fig. 5.** Position of ball (m)

In Fig. 5, with a stabilization time of 5s for both methods, LQR has a state error of 0.058m, while SMC has a state error of 0.0004m, demonstrating that the steady-state error of SMC is smaller than that of LQR, indicating that SMC is more stable further away from the equilibrium position than LQR.

From Fig. 4 and Fig. 5, LQR only stabilizes well around the equilibrium point, with the steady-state error increasing as the distance from the equilibrium position increases.

**Case 3:** The signal is set as a sine wave

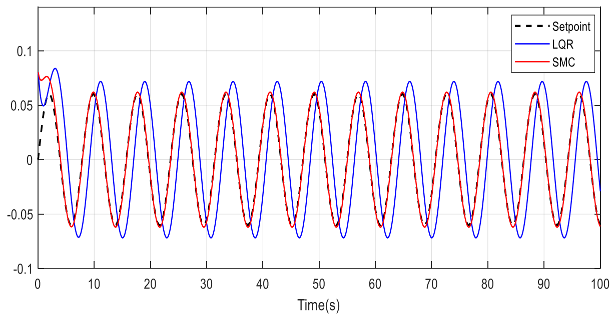

**Fig. 6.** Position of ball (m)

In Fig. 6, system under SMC closely follows reference signal better than LQR. But, LQR has a delay of approximately 1.4s.

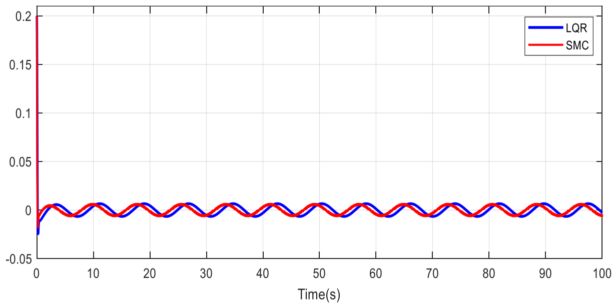

**Fig. 7**. Angle of beam (rad)

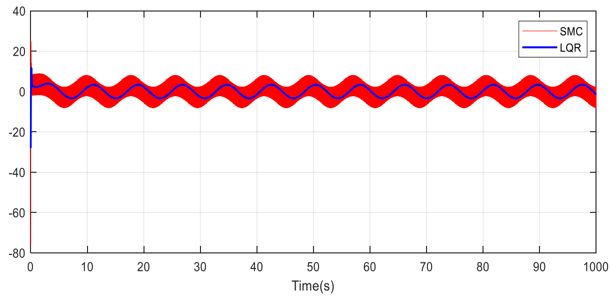

**Fig. 8.** Voltage supplied to the motor

The control algorithm of SMC involves a sign() function, causing the voltage supplied to the motor to fluctuate due to chattering phenomenon in Fig. 8, where voltage supplying to motor in SMC oscillates continuously in the range of -8.12V to 8.12V.

From observations above, linear controller LQR only stabilizes well around the equilibrium point, whereas the nonlinear SMC controller yields significantly better trajectory tracking results than linear LQR controller. This is achieved because the desired trajectory has been incorporated into design process, calculating the sliding mode. Hence, the results indicate that there is no delay when applying SMC to the system. On the other hand, LQR is a linear algorithm that applies stable control at equilibrium point. To achieve trajectory tracking control, one can only actively offset set value by subtracting a corresponding value from state variable $x_1$ before it is input of LQR controller. However, ensuring mathematical rigor for this method can be compromised, as evidenced by delay and error of the LQR controller

## 5. Experiment

An experimental model is shown in Fig. 9. The description of the image is as follows:

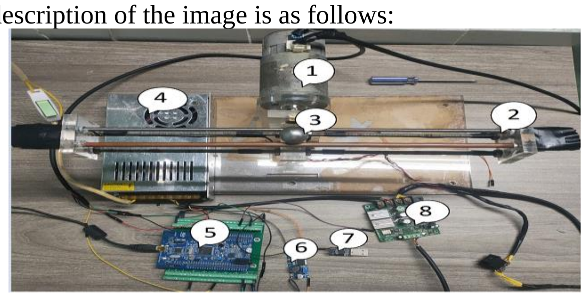


**Fig. 9.** Empirical model B&B

| | |
|---|---|
| 1.Motor DC and Encoder | 2.The beam |
| 3.The ball | 4. 12V power supply |
| 5.STM32F407 | 6.LM2596 |
| 7.CP2102 | 8. IR2184 H-bridge |

Case 1: Stable around the equilibrium position,

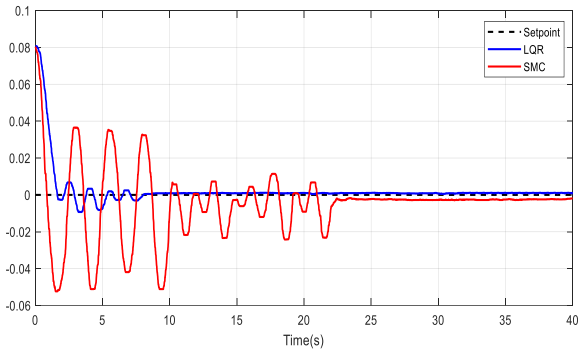


**Fig. 10.** Position of ball (m)

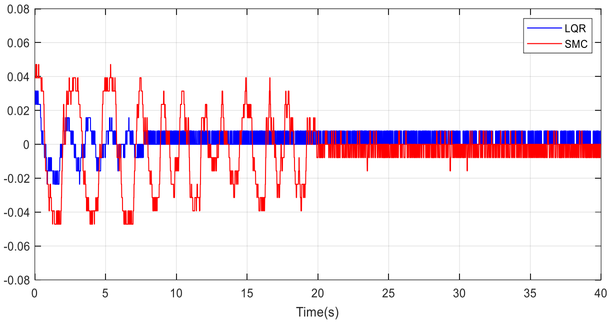


**Fig. 11.** Angle of beam (rad)

The experimental results indicate similar outcomes to the conclusions drawn from the simulation. In Fig. 10 and Fig. 11, the LQR stabilizes after approximately 8 seconds, while the SMC stabilizes after about 23s. The oscillation angle of the beam in the case of SMC ranges from -0.048 rad to 0.048rad, whereas for LQR, it oscillates between -0.024rad to 0.031rad.

**Case 2**: Worksite changes corresponding to the change in set value, $x_{1d} = 0.08(m)$

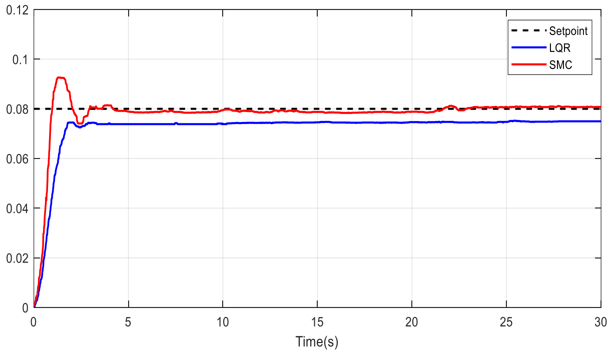


**Fig. 12.** Position of ball (m)

In Fig. 12, the state error of LQR is greater than that of SMC when tracking a setpoint 0.08m away from the equilibrium position.

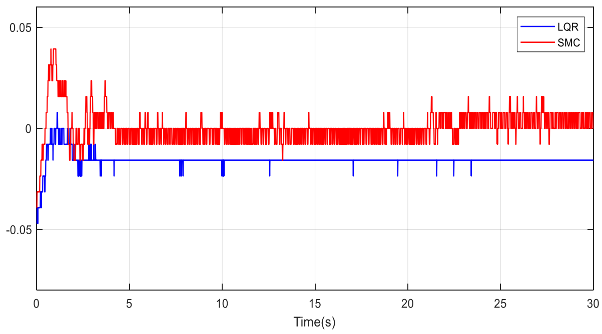


**Fig. 13.** Angle of beam (rad)

**Case 3:** The signal is set as a sine wave

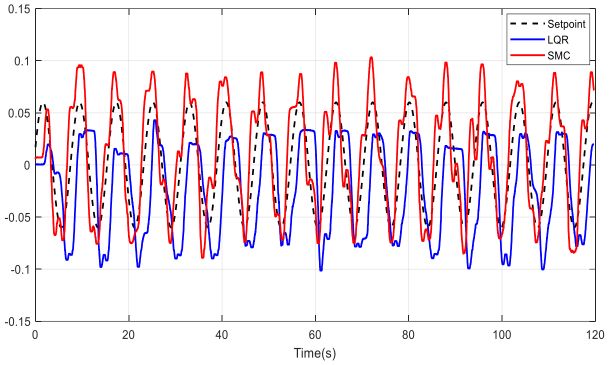


**Fig. 14.** Position of ball (m)

In Fig. 14, we observe that LQR tracks the reference signal with a delay of approximately 1.6 seconds. Differently, SMC control makes position of ball follow well the trajectory without delay time. However, the vibration of system under SMC is bigger than under LQR control.

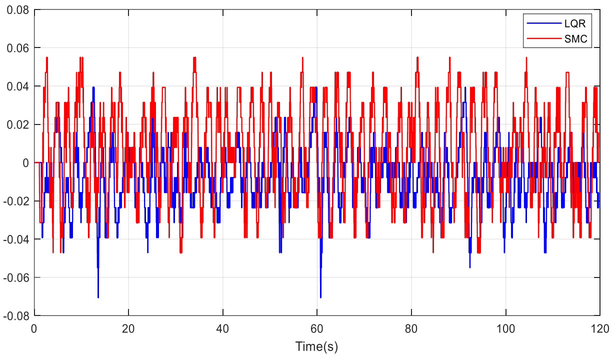


**Fig. 15.** Angle of beam (rad)

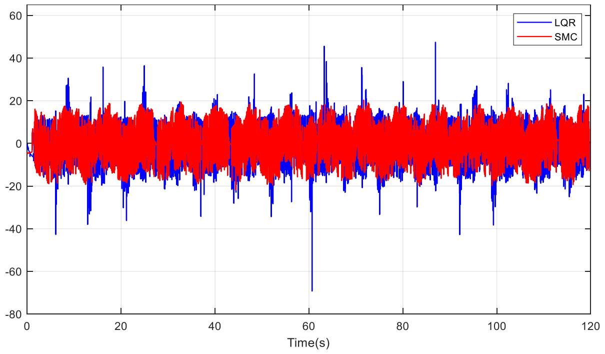


**Fig. 16**. Voltage supplied to the motor

We can see that stability of LQR is better than SMC, as SMC exhibits chattering phenomenon as shown in Fig. 16, causing continuous motor oscillations.

Thus, through the experiment, advantages and disadvantages of two control methods are shown. LQR control presents linear control stabilizes system well around the equilibrium point but less effectively at positions far from equilibrium than SMC. On the other hand, SMC represents a nonlinear controller exhibits better trajectory tracking performance compared to the nonlinear LQR controller. However, its stability is inferior to LQR due to the phenomenon of chattering.

## 6. Conclusions

In our research, by calculating and designing linear LQR and nonlinear SMC controllers for B&B system through simulation and experiment. we compare advantages, disadvantages, and control quality of two controllers. The comparison was conducted by assessing performance of both controllers in stable response around equilibrium position, tracking reference signal away from equilibrium position, and tracking a sine wave signal. We successfully designed and conducted comparative studies for both controllers. Through simulation and experimental results, non-linear SMC controller tracks reference signal better than linear LQR controller. The delay time is terminated in case of SMC but it still exists in case of LQR. However, its stability quality is inferior to LQR due to chattering phenomenon.

## 7. References

[1] Ang K.H. et al: “PID control system analysis, design, and technology”, in IEEE Transactions on Control Systems Technology, vol.13, no.4, pp. 559-576, 2005.

[2] Shah M. et al: "Control of Ball and Beam with LQR Control Scheme using Flatness Based Approach", International Conference on Computing, Electronic and Electrical Engineering (ICE Cube), 2018

[3] Sastry S., “Lyapunov Stability Theory”, Nonlinear Systems, Volume 10, Springer New York, NY, 1999.

[4] Almutairi N.B., Zribi M.: “On the sliding mode control of a ball on a beam system”, Nonlinear Dynamics, 59(1-2), 221–238, (2010).

[5] Cardona S. et al: “Sliding Mode Control based on Feedback Linearization for a Ball and Beam System”, IEEE 4th Colombian Conference on Automatic Control, 2019.

[6] Nguyen M.Q. et al: “Fuzzy Controller from Experts’ Rules for Middle Axis Ball and Beam”, Journal of fuzzy systems and control, Vol. 1, No 3, pp. 80-84, 2023.

[7] Hoang H.T.: "Intelligent control system", Ho Chi Minh City National University Publishing House, 2006.